Radial velocities vs Astrometry

- Consider an exoplanet with mass $m_p$ orbiting a star of mass $M_*$ at distance $d$, with semi-major axis $a$, period $P$, eccentricity $e$, inclination $i$ with respect to the sky.
- Radial velocity wobble:
  \[ V_r = \frac{m_p}{m_p + M_*} \frac{2\pi}{P} \frac{a \sin i}{\sqrt{1 - e^2}} (\cos (\omega + \nu) + e \cos \omega) \propto a^{-1/2} \]
  \[ K = \sqrt{\frac{2\pi G}{P} \frac{m_p}{(m_p + M_*)^{1/3}} \frac{1}{\sqrt{1 - e^2}}} \approx 28.4 \text{ m s}^{-1} \times \left( \frac{P}{1 \text{ an}} \right)^{-1/3} \left( \frac{m_p \sin i}{M_{\text{jup}}} \right)\left( \frac{M_*}{M_\odot} \right)^{-2/3} \]
- Astrometric displacement:
  \[ \Delta \alpha = \frac{m_p}{m_p + M_*} \frac{a}{d} \frac{1 - e^2}{1 + e \cos \nu} \sqrt{1 - \sin^2 (\omega + \nu) \sin^2 i} \propto \frac{m_p}{M_*} \frac{a}{d} \]
- Astrometry is **sensitive to large periods**, while velocimetry is sensitive to close-in planets.
- But it is best for **nearby stars**
Observing nearby stars

- Requirement for astrometry: be patient! (long periods)
- But for a nearby star, we can detect shorter period planets.
- Goal with such systems: Detecting solar system analogs.
- Issues to be investigated:
  - How frequent are solar-like systems around solar type stars?
  - Do they all scale the same way?
  - What about planets in binary systems?
- Dynamical issues:
  - Orbital evolution in detected systems: stability, unseen planets?
  - Perturbations by companions ⇒ Study with symplectic codes
Symplectic N-body codes (1)

• Numerical integration = finding \( q(t) \) once \( q(t-\tau) \) is known

• Classical methods do not keep \( H=Hamiltonian=cte \) (Runge-Kutta, Burlish & Stoer …)

• A symplectic scheme exactly preserves \( H \) or a nearby one

• The global error is bound

• Interesting for long term stability
Symplectic N-body codes (2)

- Basic hypotheses of symplectic integration:  
  \[ H = H_A + H_B \]  
  where we know how to exactly integrate \( H_A \) and \( H_B \). We also try to have

- On suppose en plus \( H_B << H_A \)  
  \( (H_B/H_A \sim \varepsilon) \)

- We have  
  \[ \exp(\tau A)\exp(\tau B) = \exp[\tau (A + B) + O(\tau^2)] \]

\[ \Rightarrow \] Advancing \( H_A \) during \( \tau \) and then \( H_B \) during \( \tau \), we build a first order symplectic integrator: we exactly solve a Hamiltonian

\[ H_{\text{integ}} = H + O(\tau) \]
Symplectic N-body codes (3)

• We also have

\[ \exp\left(\frac{\tau}{2}B\right)\exp(\tau A)\exp\left(\frac{\tau}{2}B\right) = \exp[\tau(A+B)+O(\tau^3)] \]

⇒ We build a second order symplectic integrator if we integrate 1) \( H_A \) during \( \tau/2 \), \( H_B \) during \( \tau \), 3) \( H_A \) during \( \tau/2 \). In this case

\[ H_{\text{integ}} = H + O(\tau^2) \]

• There are also 4, 6, 8… order methods

• In fact, if \( H_B/H_A \approx \epsilon \), we even have

\[ H_{\text{integ}} = H + O(\epsilon \tau^2) \]

⇒ We can integrate with a large timestep
Practical N-body symplectic schemes

• Simplest method: \( T + U \)
  - \( H = \) Hamiltonian of \( N \) body problem = Kinetic + Potential = \( T + U \)
  - Canonically conjugate variables: \( x_i = x, y, z, \ldots \) \( p_i = mv_x, mv_y, mv_z, \ldots \)
  - \( T = T(p_i) \) and \( U = U(x_i) \) ⇒ We can integrate separately \( T \) and \( U \)
  - ⇒ Symplectic method with \( H_A = T \), \( H_B = U \). But we don’t have \( H_B \ll H_A \)

• Second exemple: MVS method (Mixed Variable Symplectic)
  - Hypothesis: \( m_i \ll m_0 \) for every \( i = 1, \ldots, n-1 \) (planetary systems)
  - \( H_A = \) Sum of Keplerian Hamiltonians (we can integrate…)
  - \( H_B = \) What’s left = Mutual perturbations ⇒ function of \( x_i \)’s only)

• The condition \( H_B \ll H_A \) is fulfilled if
  1) Body \#0 is much more massive than the others
  2) Mutual distances are not too small (no close encounters)
Close encounters?

• Occurs when two bodies come close to each other. We have strong and short scale perturbations. Condition (2) is no longer fulfilled.

• The best to do: reducing the timestep when this occurs.

• But keeping a symplectic integrations requires a constant timestep! Two ways to solve the problem:
  – We drop symplecticity for a while during the time of the encounter (RMVS code, Levison & Duncan). Ok from a statistical point of view.
  – We change the $H_A + H_B$ splitting during the close encounter. We keep symplecticity but the computing time is longer (MERCURY code Chambers)
  – We split potentials in concentric spherical shells and attribute one timestep to each shell (SyMBA code Duncan, Lee & Levison).
The HJS method (*Hierarchical Jacobi symplectic, Beust 2003*)

- The splitting of $H$ is done according to the hierarchical structure of the stellar system.
- $H_A = \text{Sum of Keplerian Hamiltonians corresponding to centers of mass orbiting other centers of mass}$
- $H_B = \text{What’s left}$
- The condition $H_B \ll H_A$ is fulfilled irrespective of the various masses, once the nested orbits are very different in size.
Analysis of the astrometric signal

- **Issue**: detecting an astrometric wobble of a few µas
- **Perturbation sources (for a star at 50 pc)**:

<table>
<thead>
<tr>
<th>Perturbation Source</th>
<th>Effect Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proper motion over 1 period</td>
<td>0.1 µas – 1 mas</td>
</tr>
<tr>
<td>Non linear proper motion on sky</td>
<td>0.1 mas after 30 yrs</td>
</tr>
<tr>
<td>Parallactic orbit</td>
<td>20 mas</td>
</tr>
<tr>
<td>Effect of Earth eccentricity</td>
<td>0.3 mas</td>
</tr>
<tr>
<td>Barycentric motion of the Sun</td>
<td>0.15 mas</td>
</tr>
<tr>
<td>Planetary perturbations on the Earth</td>
<td>0.03 mas after 30 yrs</td>
</tr>
</tbody>
</table>

- ⇒ **Need for an accurate modeling of all these effects!**